Predictive Control for Lure Systems Subject to Constraints Using LMIs

Christoph Böhm, Shuyou Yu, Rolf Findeisen, Frank Allgöwer

Abstract— **This paper presents a computationally attractive nonlinear model predictive control approach for the class of continuous time Lure systems. The control law is obtained via the repeated solution of an efficient to solve convex optimization problem based on linear matrix inequalities (LMIs). Closed-loop stability and satisfaction of input and state constraints are guaranteed by the feasibility of the LMIs at initial time. The simulation of a flexible link robotic arm shows the applicability and effectiveness of the proposed controller.**

I. Introduction

Nonlinear model predictive control (NMPC) has received remarkable attention in the last 15 years. Its ability to deal with nonlinear control systems subject to state and input constraints makes NMPC attractive for both practical applications and theoretical research. The basic idea of standard NMPC is as follows: By solving online an optimal control problem based on the current measurement of the system states, an optimal input trajectory is obtained. The first part of this trajectory is applied to the system and the optimal control problem is solved again based on a new measurement of the system states at the next sampling instant. By now, various NMPC schemes with guaranteed stability have been developed [1, 4, 5, 7, 8, 10, 13]. However, many of those approaches suffer from two problems. Firstly, since nonlinear systems and possibly nonlinear constraints are considered, the optimization problem may be non-convex. Thus, the solution delivered by standard numerical solvers may only represent a local minima but not a global one, which can lead to a decrease in performance or even instability in the robust case. Secondly, the solution to the optimization problem often is not a state feedback law but an optimal open-loop input trajectory. The application of those open-loop trajectories in the way of standard NMPC schemes as e.g. [5, 7, 8, 13] imply that feedback (and therefore reaction on disturbances) is provided only at the sampling instants. However, in the time interval between the sampling instants the system is controlled open-loop. This requires rather short sampling intervals to counteract disturbances which may lead to computational difficulties since the optimization problem cannot be solved fast enough. The goal of this paper is to derive an NMPC method which overcomes these problems for the class of continuous time Lure systems. As in [2] and [12] the basic idea is to calculate at each sampling instant a stabilizing linear time-invariant feedback matrix via the solution of a convex optimization problem based on LMIs such that an upper bound on the considered infinite horizon cost functional is minimized. The obtained LMI conditions are similar to those derived in [1], where they are used to calculate a stabilizing terminal region and terminal penalty term. Since the approach is based on a convex optimization problem it is guaranteed that a global solution is obtained. As shown, the control strategy leads to closed-loop stability and constraint satisfaction for the set of Lure systems satisfying a sector bound on their nonlinearities. The controller is robustly stabilizing in the sense that the Lure nonlinearities can change as long as they stay sector bounded. Furthermore, the application of the feedback matrix to the system allows rather long sampling intervals since the system is controlled by a stabilizing feedback controller in the time interval between the sampling instants, in contrast to standard NMPC approaches [5, 7, 8, 13]. This makes the presented controller attractive from a computational, and thus applicational, point of view. In [14] a similar model predictive control technique has been used for the control of singular systems.

The paper is organized as follows: In Section II we discuss the considered NMPC setup and give a brief introduction to the class of Lure systems. We focus on two types of Lure systems which differ by the sector conditions their nonlinearities satisfy. Section III provides the main result of the paper, namely two stabilizing NMPC controllers based on convex optimization with guaranteed closed-loop stability and satisfaction of state and input constraints for both types of Lure systems. The presented approaches differ in aspects of applicability on one side and solvability on the other side. The simulation example of a flexible link robotic arm illustrates the obtained results in Section IV. Conclusions are provided in Section V.

II. Problem Setup

In the first part of this section we give a brief introduction to the considered class of Lure systems. The second part discusses the control task and introduces the infinite horizon cost functional which is of interest in the NMPC controller design.

A. Lure Systems

In this paper we consider a subclass of nonlinear systems, namely Lure systems, which are described by

$$\dot{x} = Ax + G\gamma(z) + Bu, \tag{1a}$$
$$z = Hx, \tag{1b}$$

(see e.g. [11]), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{n\times p}$ and $H \in \mathbb{R}^{p\times n}$ are constant linear matrices and $z \in \mathbb{R}^{p}$ denotes a

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linear combination of the states. The vector $\gamma(z): \mathbb{R}^p \to \mathbb{R}^p$ consists of p nonlinear functions depending on z. We limit our attention to Lure systems where z and the nonlinearities $\gamma(z)$ satisfy a so called sector condition of the form

$$\big(\beta z - \gamma(z)\big)^T \gamma(z) \geq 0, \tag{2}$$

where $\beta = \operatorname{diag}(\beta_1, \beta_2, \dots, \beta_p)$ with $\beta_i \in \mathbb{R}^+$, $i = 1, \dots, p$, see [11]. Figure 1 illustrates the sector condition for $p = 1$. If the constants β_i are not bounded, i.e. $\beta_i \in [0, \infty)$, the nonlinearities $\gamma(z)$ lie in the complete first and third quadrant. In this case the sector condition (2) can be formulated as

$$z^T \gamma(z) \geq 0. \tag{3}$$

Lure systems satisfying sector condition (2) possess more restricted nonlinearities compared to systems with sector condition (3). In the following we refer to (2) as growth bounded sector condition and to (3) as full sector condition. In this paper two NMPC approaches based on the online solution of a convex optimization problem subject to LMIs are derived. The first approach is applicable to Lure systems satisfying the growth bounded sector condition, the second one to systems with full sector condition. The LMIs of the first approach are less conservative. However, the second controller can be applied to a broader system class since the restrictions on the nonlinearities are weaker.

B. *Control Task*

The control task considered is to stabilize the origin of (1) such that polytopic state and input constraints of the form

$$\mathscr{C} = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m} : c_i x + d_i u \leq 1, i = 1, \cdots, r \right\} \tag{4}$$

are satisfied. For the solution of this task we propose the use of NMPC. At each sampling instant t_k the NMPC controller minimizes the infinite horizon cost functional

$$J\big(x(\cdot), u(\cdot)\big) = \int_{t_k}^{\infty} x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \, \mathrm{d}\tau \tag{5}$$

with the positive definite weighting matrices $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$, subject to the constraints (4) and the system dynamics (1). Due to the infinite horizon, the resulting optimization problem is in general not solvable. Therefore, in standard NMPC approaches a large but finite prediction horizon is chosen instead in order to obtain a satisfying approximation of the infinite horizon cost. By numerical solvers the resulting finite horizon optimal control problem is solved repeatedly to obtain the optimal input. However, to calculate the optimal open-loop input trajectory, often still high computational effort is necessary. Furthermore, since we consider nonlinear systems the optimization problem might be non-convex. To overcome these problems, the basic idea of this paper is to calculate at each sampling instant t_k a feedback matrix K_k instead of a complete input trajectory. Thus, the resulting input applied to system (1) is of the form

$$u(t) = K_k x(t), \ t \in [t_k, t_{k+1}). \tag{6}$$

Here, t_k and t_{k+1} denote consecutive sampling instants at which the optimal control problem is solved. Following the ideas of [2] and [12], the feedback matrix K_k is calculated such that an upper bound on the infinite horizon cost functional (5) is minimized at each sampling instant t_k via the solution of a convex optimization problem based on LMIs. This method offers several advantages compared to standard NMPC approaches. First, the considered optimization problem is convex. Thus, it is guaranteed that a global minima is obtained if the optimization problem is feasible. Moreover, the solution to the optimization problem can be obtained significantly faster. As a further advantage, the application of the feedback matrix K_k allows that the sampling interval $\delta = t_{k+1} - t_k$ can be chosen larger than in standard NMPC, since the system is controlled in closed-loop also between the sampling instants. This makes the proposed method appealing and suitable also for systems with rather small time constants. Finally, the novel controller robustly stabilizes the considered system if the nonlinearities are only known to be sector bounded, whereas classical NMPC approaches need exact knowledge about the nonlinear dynamics in order to predict the system behaviour.

In the following section we present the main result of this paper, namely two NMPC controllers based on a convex optimization problem involving LMIs. The first controller is applicable to Lure systems satisfying a growth bounded sector condition (2), the second one to Lure systems with full sector condition (3).

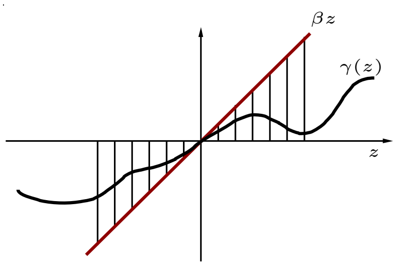

Fig. 1. Sector condition of Lure systems.

III. MAIN RESULTS

The two controllers derived in this section rely on the following Lemma to guarantee satisfaction of state and input constraints as defined in (4).

Lemma 1: The ellipsoid $\mathscr{D} = \{y \in \mathbb{R}^n : y^T F y \leq \mu\}$ is contained in the set $\mathscr{W} = \{y \in \mathbb{R}^n : w_i y \leq 1, \ i = 1, \dots, r\}$, where $F \in \mathbb{R}^{n \times n}$ and $w_i \in \mathbb{R}^{1 \times n}$, if and only if

$$w_i (\mu F^{-1}) w_i^T \leq 1, \ i = 1, \dots, r. \tag{7}$$

Proof: See [3] and [6]. ■

A. *Lure systems with growth bounded sector condition*

In the following we consider Lure systems satisfying the growth bounded sector condition (2). As illustrated in Figure 1 the nonlinearities lie in the first and third quadrant but are growth bounded. Inequality (2) can be expressed in matrix form by

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2} H^T \beta^T \\ -\frac{1}{2} \beta H & I \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq 0. \tag{8}$$

The following lemma provides a condition to calculate a stabilizing feedback law for system (1) with sector condition (2) and an upper bound on the infinite horizon cost functional (5).

Lemma 2: Consider the system (1) with the growth bounded sector condition (8). Suppose that there exist matrices $0 < \Lambda = \Lambda^T \in \mathbb{R}^{n\times n}$ and $\Gamma \in \mathbb{R}^{m\times n}$, and constants $\tau \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$ such that the matrix inequality

$$\begin{bmatrix} -\Delta-\Delta^T & -S & \Lambda Q^{\frac{1}{2}} & \Gamma^T R^{\frac{1}{2}} \\ -S^T & \alpha\tau I & 0 & 0 \\ Q^{\frac{1}{2}}\Lambda & 0 & \alpha I & 0 \\ R^{\frac{1}{2}}\Gamma & 0 & 0 & \alpha I \end{bmatrix} > 0 \qquad (9)$$

is satisfied with $\Delta = [A\ B][\Lambda\ \Gamma^T]^T$ and $S = G\alpha + \frac{\tau}{2}\Lambda H^T\beta^T$. Then for $P = \alpha\Lambda^{-1}$ and $K = \Gamma\Lambda^{-1}$ the following holds:

a. The feedback law $u = Kx$ asymptotically stabilizes system (1) with sector condition (2).
b. $\bar{V} = x^T(t)Px(t)$ is an upper bound on the infinite horizon cost functional at time t, where $x(t)$ is the state of system (1) at time t.

Proof: *Applying the Schur complement to (9), substituting Λ, Γ, S and Δ as defined in the lemma and multiplying the obtained inequality from both sides with diag(P,I), we know that the inequality*

$$\begin{bmatrix} \begin{bmatrix} A^TP+PA+K^TB^TP \\ +PBK+Q+K^TRK \end{bmatrix} & PG+\frac{\tau}{2}H^T\beta^T \\ G^TP+\frac{\tau}{2}\beta H & -\tau I \end{bmatrix} < 0 \qquad (10)$$

is satisfied. Applying the $\mathscr{S}$-procedure, see e.g. [3], it follows that

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} A^TP+PA+K^TB^TP \\ +PBK+Q+K^TRK \end{bmatrix} & PG \\ G^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} < 0 \qquad (11)$$

holds for all x and γ satisfying (8). This is equivalent to

$$x^T(A^TP+PA+K^TB^TP+PBK+Q+K^TRK)x + xPG\gamma(z)+\gamma^T(z)G^TPx < 0. \qquad (12)$$

With $V(x) = x^TPx > 0$, $Q > 0$ and $R > 0$ we know that

$$\dot{V}(x) = x^T(A^TP+PA+K^TB^TP+PBK)x + x^TPG\gamma(z)+\gamma^T(z)G^TPx < 0 \qquad (13)$$

is satisfied. Thus, clearly $V(x) = x^TPx$ is a Lyapunov function and the control law $u = Kx$ asymptotically stabilizes system (1), which proofs part (a) of Lemma 2.
Integrating inequality (12) from $\tau = t$ to $\tau \to \infty$, with $u = Kx$ (leading to $x(t) = 0$ for $t \to \infty$) we obtain

$$\bar{V} = x^T(t)Px(t) > \int_t^\infty x^T(\tau)Qx(\tau)+u^T(\tau)Ru(\tau)d\tau. \qquad (14)$$

Thus, $\bar{V}$ is an upper bound on the infinite horizon cost functional (5), which proofs part (b) of the lemma. ■

Note that so far no constraint satisfaction is guaranteed. Furthermore, the controller might be quite conservative if the system is far away from the origin. To guarantee less conservativeness we propose to calculate the feedback matrix repeatedly at the sampling instants t_k. Therefore, in the following the index k describes the association of matrices, optimization variables, functions etc. with the time instant t_k. Using the results of Lemma 2 in the following theorem we propose an NMPC controller with guaranteed stability and constraint satisfaction that minimizes an upper bound on the cost functional (5) at each recalculation time instant t_k.

Theorem 1: Consider the system (1) with the growth bounded sector condition (8). The NMPC controller given by the repeated solution of the optimization problem

$$\min_{\alpha_k,\Lambda_k,\Gamma_k} \alpha_k \qquad (15a)$$

subject to

$$\begin{bmatrix} 1 & x^T(t_k) \\ x(t_k) & \Lambda_k \end{bmatrix} > 0, \qquad (15b)$$

$$\begin{bmatrix} -\Delta_k-\Delta_k^T & -S_k & \Lambda_k Q^{\frac{1}{2}} & \Gamma_k^T R^{\frac{1}{2}} \\ -S_k^T & \alpha_k\tau I & 0 & 0 \\ Q^{\frac{1}{2}}\Lambda_k & 0 & \alpha_k I & 0 \\ R^{\frac{1}{2}}\Gamma_k & 0 & 0 & \alpha_k I \end{bmatrix} > 0, \qquad (15c)$$

$$\begin{bmatrix} 1 & c_i\Lambda_k+d_i\Gamma_k \\ (c_i\Lambda_k+d_i\Gamma_k)^T & \Lambda_k \end{bmatrix} \geq 0, \qquad (15d)$$
$$i = 1,\ldots,r,$$

at the sampling instants t_k based on the state $x(t_k)$, with $P_k = \alpha_k\Lambda_k^{-1}$ and $K_k = \Gamma_k\Lambda_k^{-1}$ has the following properties:

a. The optimization problem (15) is convex if τ is fixed. Furthermore, it is feasible at the sampling instant t_{k+1} if it is feasible at t_k.
b. The solution to the optimization problem (15) minimizes the upper bound $\bar{V}_k = x^T(t_k)P_kx(t_k)$ on the cost functional (5) at each sampling instant t_k.
c. If the optimization problem (15) is feasible at $t_0 = 0$, the control law

$$u(t) = K_kx(t),\ t \in [t_k, t_{k+1}), \qquad (16)$$

asymptotically stabilizes the origin of the system (1) with the growth bounded sector condition (8), and the input and state constraints (4) are satisfied for all times $t \geq 0$.

Proof: *The proof is divided into three parts establishing the properties (a)-(c).*
Part(a): If τ is fixed, inequalities (15b)-(15d) are LMIs and therefore convexity of the optimization problem (15) follows trivially. Since only (15b) depends on $x(t_k)$, clearly the solution to the optimization problem (15) at the sampling instant t_k also satisfies the LMIs (15c) and (15d) at the sampling instant t_{k+1}. Furthermore, (15c) is identical to (9) in Lemma 2. Therefore, it follows from (13) that

$$x^T(t_{k+1})P_kx(t_{k+1}) < x^T(t_k)P_kx(t_k). \qquad (17)$$

Applying the Schur complement to (15b), substituting Λ_k, and combining the obtained inequality with (17), it follows *that the inequality*

$$x^T(t_{k+1})P_k x(t_{k+1}) < x^T(t_k)P_k x(t_k) < \alpha_k \tag{18}$$

is satisfied for all t_k. Thus, the solution to the optimization problem at the sampling instant t_k also satisfies condition (15b) at the sampling instant t_{k+1}. By induction it follows that feasibility at $t_0 = 0$ implies feasibility at all future sampling instants.

Part (b): From Lemma 2 we know that $\overline{V}_k = x^T(t_k)P_k x(t_k)$ is an upper bound on the cost functional (5). It follows from the proof of part (a) that

$$\overline{V}_k = x^T(t_k)P_k x(t_k) < \alpha_k. \tag{19}$$

Thus, minimizing α_k implies the minimization of the upper bound $\overline{V}_k$, see [2, 12] for details.

Part (c): In order to show stability we consider the candidate jump Lyapunov functional $\mathscr{V}_k(x) = x^T P_k x$, where P_k is recalculated at each sampling instant t_k. From the proof of Lemma 2 it follows that the application of the control law (16) leads to $\dot{\mathscr{V}}_k < 0 \ \forall\ t \in [t_k, t_{k+1})$. If it can be shown that furthermore at the sampling instants

$$\mathscr{V}_{k+1}\big(x(t_{k+1})\big) \le \mathscr{V}_k\big(x(t_{k+1})\big) \tag{20}$$

is satisfied, Lyapunov stability of the system (1) can be deduced. We know from the proof of part (a) that the solution to the optimization problem at t_k is a feasible, however in general suboptimal, solution at t_{k+1}. Thus, it follows that

$$x^T(t_{k+1})P_{k+1}x(t_{k+1}) \le x^T(t_{k+1})P_k x(t_{k+1}) \tag{21}$$

is satisfied at each sampling instant t_{k+1}, which is equivalent to (20). To conclude the proof, it remains to establish the constraint satisfaction. Substituting the feedback control law $u = K_k x$ in (4) we obtain the constraint set

$$\mathscr{C}_k = \big\{x \in \mathbb{R}^n : (c_i + d_i K_k)x \le 1,\ i = 1\ldots,r\big\}, \tag{22}$$

which due to the state feedback in the control law only depends on the system state x. We now show that the ellipsoid

$$\mathscr{E}_k = \big\{x \in \mathbb{R}^n : x^T P_k x \le \alpha_k\big\} \tag{23}$$

lies in the constraint set (22) at the sampling instant t_k and moreover is an invariant set under the control law (16). Applying the Schur complement to (15d) and substituting P_k, K_k and α_k as defined in the theorem, we obtain

$$1 - (c_i + d_i K_k)P_k^{-1}\alpha_k(c_i + d_i K_k)^T \ge 0,\ i = 1,\ldots,r. \tag{24}$$

Thus, using Lemma 1 it follows from (15d) that the ellipsoid $\mathscr{E}_k$ is contained in the constraint set (22). Consequently, all states lying in the ellipsoid satisfy state and input constraints. Clearly, since (19) holds, the system state $x(t_k)$ lies in the ellipsoid at time instant t_k. Therefore, with the matrix P_k being constant in the time interval $t \in [t_k, t_{k+1})$, state and input constraints are satisfied in this interval if the ellipsoid $\mathscr{E}_k$ is an invariant set under the control law (16). From the proof of Lemma 2 we know that $V_k = x^T P_k x$ is a Lyapunov function for system (1). From this follows that the inequality

$$x^T(t)P_k x(t) < x^T(t_k)P_k x(t_k) < \alpha_k \tag{25}$$

is satisfied for each $t \in (t_k, t_{k+1})$. Therefore, all states in the time interval $[t_k, t_{k+1})$ lie in the ellipsoid $\mathscr{E}_k$, which guarantees satisfaction of state and input constraints. Since this can be shown for each sampling instant t_k, it follows that the control law (16) is such that the constraints are satisfied for all times $t \ge 0$. ■

Inequality (15c) is only an LMI if τ is considered fixed and not as an optimization variable. Although a reasonable fixed value for τ can be determined off-line, this increases the conservativeness of the presented NMPC approach, with drawbacks concerning feasibility and performance. However, numerous simulations suggest that the NMPC controller and the resulting performance is only weakly sensitive towards τ. Nevertheless, it is reasonable to discuss solutions which overcome this problem. In the following subsection we present a second NMPC approach for Lure systems with a less restrictive sector condition.

B. Lure systems with full sector condition

The approach presented in this subsection considers Lure systems (1) satisfying the full sector condition (3) which requires that the nonlinearities $\gamma(z)$ lie in the complete first and complete third quadrant. Thus, the nonlinearities are less growth bounded. This implies that the approach presented in the following is applicable to a broader system class.

Similar to Lemma 2 the following lemma gives conditions for the calculation of a stabilizing feedback law for system (1) and of an upper bound on the cost functional (5).

Lemma 3: Consider the system (1) subject to the full sector condition (3). Suppose that there exist matrices $0 < \Lambda = \Lambda^T \in \mathbb{R}^{n\times n}$ and $\Gamma \in \mathbb{R}^{m\times n}$ and a constant $\alpha \in \mathbb{R}^+$ such that the LMI

$$\begin{bmatrix} -\Delta - \Delta^T & \Lambda Q^{\frac{1}{2}} & \Gamma^T R^{\frac{1}{2}} \\ Q^{\frac{1}{2}}\Lambda & \alpha I & 0 \\ R^{\frac{1}{2}}\Gamma & 0 & \alpha I \end{bmatrix} > 0 \tag{26}$$

and the equality constraint

$$-H\Lambda = \alpha G^T \tag{27}$$

are satisfied with $\Delta = [A\ B][\Lambda\ \Gamma^T]^T$. Then for $P = \alpha\Lambda^{-1}$ and $K = \Gamma\Lambda^{-1}$ the following holds:

a. The feedback law $u = Kx$ asymptotically stabilizes the system (1) under the full sector condition (3).
b. $\overline{V} = x^T(t)Px(t)$ is an upper bound on the infinite horizon cost functional (5) at time t, where $x(t)$ is the state of the system (1) at time t.

Proof: *By using the same arguments as in the proof of Lemma 2 we can show that both properties (a) and (b) hold if the LMI (26) and the equality constraint (27) imply that the inequality (12) is satisfied. This is clearly the case if we can show that both the matrix inequality*

$$PA + A^T P + PBK + K^T B^T P + Q + K^T RK < 0 \tag{28}$$

and the inequality

$$x^T PG\gamma(z) + \gamma^T(z)G^T Px < 0 \tag{29}$$

are satisfied. Substituting Λ in (27) it follows that the equality $-H = G^T P$ holds. Plugging this into (29) with (1b) we obtain

$$-z^T\gamma(z) - \gamma^T(z)z < 0, \tag{30}$$

which is clearly satisfied due to the sector condition (3). Thus, inequality (29) holds. Furthermore, by applying the Schur complement on (26) and substituting Λ and Γ we obtain (28). Thus, (26) and the equality constraint (27) imply satisfaction of (12), which according to the proof of Lemma 2 guarantees that the properties (a) and (b) hold. ■

With the results of Lemma 3 we can formulate the following theorem for an NMPC controller for Lure systems satisfying the full sector condition (3).

Theorem 2: Consider the system (1) with the full sector condition (3). The NMPC controller given by the repeated solution of the optimization problem

$$\min_{\alpha_k,\Lambda_k,\Gamma_k} \alpha_k \tag{31a}$$

subject to

$$\begin{bmatrix} 1 & x^T(t_k) \\ x(t_k) & \Lambda_k \end{bmatrix} > 0, \tag{31b}$$

$$\begin{bmatrix} -\Delta_k - \Delta_k^T & \Lambda_k Q^{\frac{1}{2}} & \Gamma_k^T R^{\frac{1}{2}} \\ Q^{\frac{1}{2}}\Lambda_k & \alpha_k I & 0 \\ R^{\frac{1}{2}}\Gamma_k & 0 & \alpha_k I \end{bmatrix} > 0, \tag{31c}$$

$$\alpha_k G^T + H\Lambda_k = 0 \tag{31d}$$

$$\begin{bmatrix} 1 & c_i\Lambda_k + d_i\Gamma_k \\ (c_i\Lambda_k + d_i\Gamma_k)^T & \Lambda_k \end{bmatrix} \geq 0, \quad i = 1,\ldots,r, \tag{31e}$$

at the sampling instants t_k based on the state $x(t_k)$, with $P_k = \alpha_k\Lambda_k^{-1}$ and $K_k = \Gamma_k\Lambda_k^{-1}$ has the following properties:

a. The optimization problem (31) is convex. Furthermore, it is feasible at the sampling instant t_{k+1} if it is feasible at t_k.
b. The solution to the optimization problem (31) minimizes the upper bound $\bar{V}_k = x^T(t_k)P_k x(t_k)$ on the cost functional (5) at each sampling instant t_k.
c. If the optimization problem (31) is feasible at $t_0 = 0$, the control law

$$u(t) = K_k x(t),\ t \in [t_k, t_{k+1}), \tag{32}$$

asymptotically stabilizes the origin of the system (1) with the full sector condition (3) and the input and state constraints (4) are satisfied for all times $t \geq 0$.

Proof: The proof uses the results of Lemma 3 and the same arguments as in the proof of Theorem 1. ■

The NMPC controller proposed by Theorem 2 overcomes the convexity problem with respect to τ present in Theorem 1. However, the relaxed sector condition makes the approach in general more conservative. Thus, one has to decide for each specific problem which of the proposed NMPC controllers is more suitable to satisfy the control task.

IV. Simulation Example

To illustrate the performance of the controller derived for the growth bounded sector condition we consider the dynamics of a flexible link robotic arm (see e.g. [1, 9]) which are given by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -16.7 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \tag{33}$$

$$G^T = \begin{bmatrix} 0 & 0 & 0 & -3.33 \end{bmatrix},\ H = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

and the nonlinearity

$$\gamma(z) = \sin z + z. \tag{34}$$

To fulfill the sector condition (2) the inequality

$$\left(\beta z - \sin(z) - z\right)^T \left(\sin(z) + z\right) \geq 0. \tag{35}$$

has to hold. This is the case for all $\beta \geq 2$. In order to reduce conservativeness for the simulation we have chosen the smallest possible value $\beta = 2$. The constraint set $\mathscr{C}$ is defined by the input constraints $-1.5 \leq u \leq 1.5$ and the state constraints $-\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq x_3 \leq \frac{\pi}{2}$. The control task is to steer the robotic arm to the origin with the NMPC controller derived in Theorem 1. For the simulation we have chosen the weighting matrices

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \ R = 0.1. \tag{36}$$

Off-line calculations have shown that $\tau = 1$ is a suitable choice to obtain good controller performance. The black lines in Figure 2 show the simulation results obtained by the proposed NMPC controller starting from $x_0 = [1.2\ 0\ 0\ 0]^T$. To illustrate the effectiveness of this approach we compare the results with those obtained when the static control law calculated at the first sampling instant is applied to the system without recalculating the feedback matrix. Clearly, the NMPC controller steers the system to the origin much faster. The update of the control law allows to exploit the available input energy more efficiently. This leads to a more aggressive input trajectory and a faster controller performance.

V. Conclusions

In this paper two computationally attractive NMPC controllers for two types of Lure systems, which differ by the sector condition their nonlinearities satisfy, have been derived. In both approaches the control law is the solution to a convex optimization problem based on linear matrix inequalities that is solved repeatedly at each sampling instant. The obtained solutions minimize an upper bound on the considered infinite horizon cost functional. If the optimization problem is initially feasible, both controllers guarantee closed-loop stability and satisfaction of state and input constraints. The effectiveness of the presented results have been illustrated by a simulation of a flexible link robotic arm.

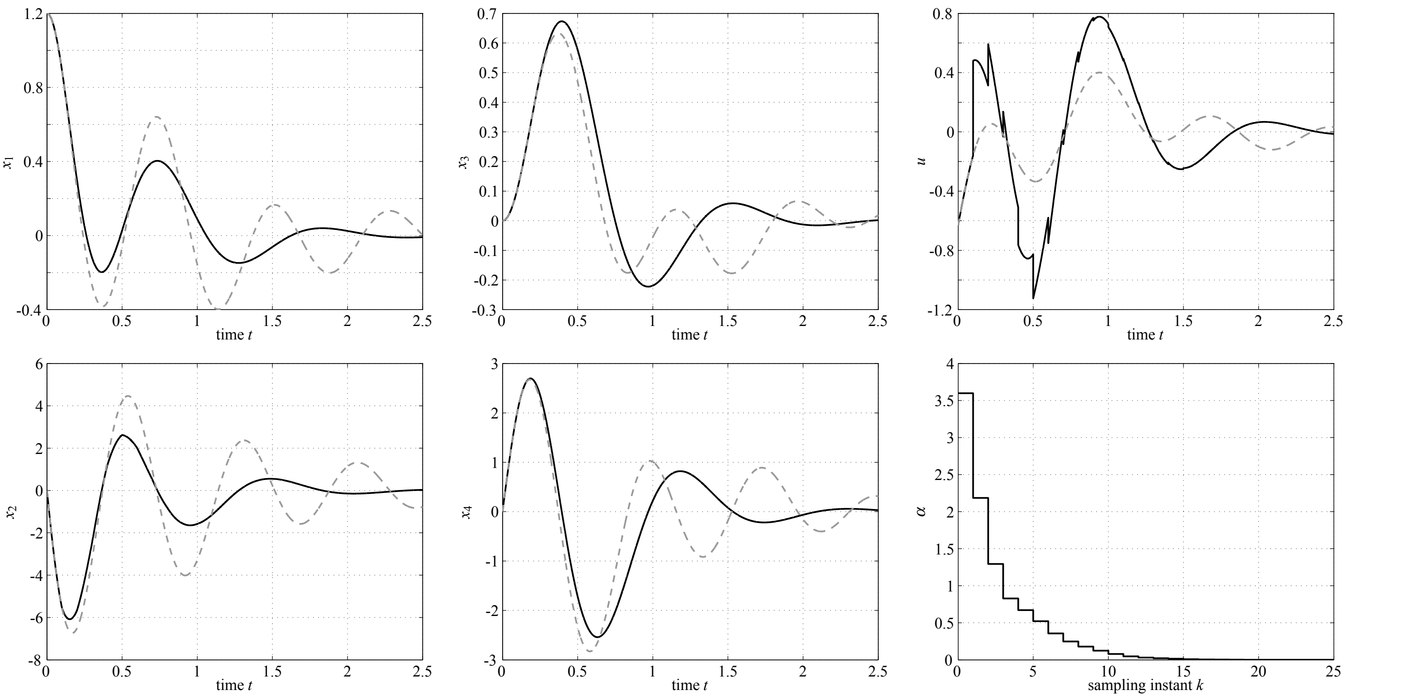

Fig. 2. Comparison of the proposed NMPC controller (black solid line) with the static controller calculated at the first sampling instant (gray dashed line). Four plots on the left part: States x of the robotic arm. Upper right plot: Input u. Lower right plot: Upper bound on the considered cost functional α.

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